

IDEMPOTENT BONDING RELATIONS ARE NONTRIVIAL IF AND ONLY IF THEY SATISFY CONDITION Γ

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ABSTRACT. A relation $f \subseteq X^2$ satisfies condition Γ if there exist distinct $x, y \in X$ with $\langle x, x \rangle, \langle x, y \rangle, \langle y, y \rangle \in f$. The authors improve a previous result by characterizing nontrivial idempotent bonding relations on compact Hausdorff spaces as those satisfying condition Γ .

1. PRELIMINARIES

Assume X is always a compact Hausdorff space.

Let a relation $f \subseteq X^2$ be full if $\forall x \in X \exists y \in X (\langle x, y \rangle \in f)$. We define a bonding relation $f \subseteq X^2$ on X to be a full relation which is a closed subset of X^2 . Such relations are often alternately characterized as upper-semicontinuous (u.s.c.) maps, which are continuous functions from X to the space $H(X)$ of nonempty closed subsets of X . As such let $f(x) = \{y \in X : \langle x, y \rangle \in f\}$, $f[A] = \{y \in X : \exists x \in A (\langle x, y \rangle \in f)\}$, and $f^2 = f \circ f = \{\langle x, z \rangle \in X^2 : \exists y \in X (\langle x, y \rangle, \langle y, z \rangle \in f)\}$, that is, $f^2(x) = f[f(x)]$.

A relation is idempotent if $f = f^2$. It is surjective if for each $y \in X$, there exists $x \in X$ where $\langle x, y \rangle \in f$. For $A \subseteq X$, let $f \upharpoonright A = \{\langle x, y \rangle \in f : x \in A\}$ be the restriction of f to A . Note that if f is idempotent then $f \upharpoonright f(x)$ is surjective (onto $f(x)$) for all $x \in X$. Let $\iota = \{\langle x, x \rangle : x \in X\}$ be the identity relation. We say a bonding relation f is nontrivial if for some $x \in X$, $f \upharpoonright f(x) \neq \iota \upharpoonright f(x)$. A single-valued bonding relation satisfies $|f(x)| = 1$ for all $x \in X$.

It's important to note that if f is an idempotent surjective single-valued bonding relation, then $f = \iota$ and thus is trivial. Likewise, every trivial idempotent surjection is the single-valued identity.

However there are trivial idempotent bonding relations besides the identity: take for instance $t \subseteq \{0, 1, 2\}^2$ defined by $t = \{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 0 \rangle, \langle 2, 1 \rangle\}$. Then $t \upharpoonright f(2) = t \upharpoonright \{0, 1\} = \iota$; of course, t fails to map to 2 and is not surjective. By connecting the dots the reader may sketch a version of t defined for the closed interval $[0, 2] \subseteq \mathbb{R}$.

Say that f satisfies condition Γ if there exist distinct $x, y \in X$ such that $\langle x, x \rangle, \langle x, y \rangle, \langle y, y \rangle \in f$. The authors will show that an idempotent bonding relation is nontrivial if and only if it satisfies condition Γ . This note answers their question in [1] by generalizing their result on interval-valued idempotent relations defined on the closed interval $[0, 1] \subseteq \mathbb{R}$.

2. MAIN RESULT

Lemma 1. *Every nontrivial idempotent bonding relation f contains two points $\langle x, x \rangle$ and $\langle y, x \rangle$ for distinct $x, y \in X$.*

Proof. Note first that if $\iota \subsetneq f$, then the lemma follows immediately. So let $x_0 \in X$ be a point where $\langle x_0, x_0 \rangle \notin f$.

Suppose x_i is defined for $i \leq n$ such that $\langle x_i, x_j \rangle \in f$ if and only if $i < j$. So we may choose x_{n+1} distinct from x_i for $i \leq n$ such that $\langle x_n, x_{n+1} \rangle \in f$. If $\langle x_{n+1}, x_{n+1} \rangle \in f$, then the lemma is satisfied by $x = x_{n+1}$ and $y = x_n$. Note that by idempotence, $\langle x_{n+1}, x_i \rangle \notin f$ for $i \leq n$ as otherwise $x_i \in f(x_{n+1}) \subseteq f(f(x_n)) = f(x_n)$ contradicting $\langle x_n, x_i \rangle \notin f$.

Since $\{x_n : n < \omega\}$ is an infinite set in a compact Hausdorff space, it has a limit point x_ω . Note then that for any open neighborhood U of x_ω , U contains infinitely many x_n , so choose $i < j$ such that $x_i, x_j \in U$. Then, it follows that the basic open neighborhood U^2 of $\langle x_\omega, x_\omega \rangle$ contains $\langle x_i, x_j \rangle$. Thus $\langle x_\omega, x_\omega \rangle$ is a limit point of $\{\langle x_i, x_j \rangle : i < j < \omega\} \subseteq f$, and as f is closed, $\langle x_\omega, x_\omega \rangle$ belongs to f . Then since $x_\omega \neq x_0$ (as $\langle x_0, x_0 \rangle \notin f$), we may similarly show $\langle x_0, x_\omega \rangle$ is a limit point of $\{\langle x_0, x_n \rangle : 0 < n < \omega\} \subseteq f$, and therefore $\langle x_0, x_\omega \rangle \in f$. The lemma is now witnessed by $x = x_\omega$ and $y = x_0$. \square

Lemma 2. *Suppose $\langle x, x \rangle, \langle x, y \rangle \in f$ for distinct $x, y \in X$ and an idempotent bonding relation f . Then f satisfies condition Γ .*

Proof. Let $z_0 = y$. If $\langle y, y \rangle = \langle z_0, z_0 \rangle \in f$, we are done.

Suppose z_i is defined for $i \leq n$ such that $\langle z_i, z_j \rangle \in f$ if and only if $i < j$, and $\langle x, z_i \rangle \in f$ for $i \leq n$. So we may choose z_{n+1} distinct from z_i for $i \leq n$ such that $\langle z_n, z_{n+1} \rangle \in f$. Note that $\langle x, z_{n+1} \rangle \in f$ since $\langle x, z_n \rangle, \langle z_n, z_{n+1} \rangle \in f$ and thus $z_{n+1} \in f(z_n) \subseteq f(f(x)) = f(x)$. If $\langle z_{n+1}, z_{n+1} \rangle \in f$, then the condition Γ is witnessed by $\langle x, x \rangle, \langle x, z_{n+1} \rangle, \langle z_{n+1}, z_{n+1} \rangle$. On the other hand, $\langle z_{n+1}, z_i \rangle \notin f$ for $i \leq n$ as otherwise by idempotence $z_i \in f(z_{n+1}) \subseteq f(f(z_n)) = f(z_n)$ contradicting $\langle z_n, z_i \rangle \notin f$. Similarly, $\langle z_n, x \rangle \notin f$ as otherwise $\langle z_n, x \rangle, \langle x, z_n \rangle \in f \Rightarrow \langle z_n, z_n \rangle \in f$.

Since $\{z_n : n < \omega\}$ is an infinite set in a compact Hausdorff space, it has a limit point z . Note then that for any open neighborhood U of z , U contains infinitely many z_n , so choose $i < j$ such that $z_i, z_j \in U$. Then, it follows that the basic open neighborhood U^2 of $\langle z, z \rangle$ contains $\langle z_i, z_j \rangle$. Thus $\langle z, z \rangle$ is a limit point of $\{\langle z_i, z_j \rangle : i < j < \omega\} \subseteq f$, and as f is closed, $\langle z, z \rangle$ belongs to f . We may similarly show $\langle x, z \rangle$ is a limit point of $\{\langle x, z_n \rangle : 0 < n < \omega\} \subseteq f$, and therefore $\langle x, z \rangle \in f$.

We know $x \neq z$ since otherwise $\{\langle z_0, z_{n+1} \rangle : n < \omega\} \subseteq f$ would imply its limit $\langle z_0, z \rangle = \langle z_0, x \rangle \in f$, which was disproved above. Therefore $\langle x, x \rangle, \langle x, z \rangle, \langle z, z \rangle \in f$ witness condition Γ . \square

Lemma 3. *The inverse of an idempotent relation is also an idempotent relation.*

Proof. $(f^{-1})^2 = (f^2)^{-1} = f^{-1}$. \square

Theorem 4. *An idempotent bonding relation is nontrivial if and only if it satisfies condition Γ if and only if it contains two points $\langle x, x \rangle, \langle x, y \rangle$.*

Proof. Obviously, if a bonding relation f has condition Γ then it contains two points $\langle x, x \rangle, \langle x, y \rangle$. It then follows from those two points that $f \restriction f(x)$ is not the identity, and therefore f is nontrivial.

If f is nontrivial idempotent, then apply Lemma 1 to obtain the points $\langle x, x \rangle, \langle y, x \rangle \in f$. Then $\langle x, x \rangle, \langle x, y \rangle \in f^{-1}$, which is idempotent by Lemma 3. So Lemma 2 may be applied to show that f^{-1} has condition Γ , and therefore so does f . \square

3. AN APPLICATION

The authors used f 's condition Γ in [1] to show that for an ordinal α , the inverse limit $\varprojlim\{I, f, \alpha\}$ is metrizable if and only if α is countable. Using a few unpublished results of the first author along with the main result of this note, this in fact generalizes to the following theorem:

Theorem 5. *Let f be a nontrivial bonding relation on a compact metrizable space X , and let L be an arbitrary total order. Then the inverse limit $\varprojlim\{X, f, L\}$ is metrizable if and only if it is Corson compact if and only if L is countable.*

Proof. If L is countable then the subspace $\varprojlim\{X, f, L\}$ of the metrizable space X^L is of course metrizable. If L is uncountable, note that $\varprojlim\{X, f, L\}$ contains the subspace $\varprojlim\{2, \gamma, L\}$ where $\gamma = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 1 \rangle\}$. It may be shown that $\varprojlim\{2, \gamma, L\}$ is homeomorphic to a compact linearly ordered topological space \tilde{L} which is metrizable if and only if it is Corson compact if and only if it is second-countable. The result follows by showing that \tilde{L} is second-countable if and only if L is countable. \square

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